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Note

About the Time Evolving Voronoi Tessellation

1. INTRODUCTION

Voronoi diagrams are relevant in different areas of physics such as condensed non-crystalline systems, including liquids, metallic glasses, and oxide glasses [1–2]. Their construction depends strongly on the algorithm one uses, which can range from classical geometrical methods to construct the vertices of the polyhedra [3] to more complicated procedures that make use of various lemmas and theorems to extract the number of faces [4–6].

All these algorithms are not very easy to handle; we therefore decided to use more simple arguments on a finite lattice. These have already been introduced by Kiang [7] who used a 80×80 point lattice covering a square and a $20 \times 20 \times 20$ point lattice filling a cube. Now due to the great improvement in the computing facilities we can handle a $400 \times 400 \times 400$ cubic lattice.

In Ref. [7] the size distribution of random Voronoi segments has also been analyzed and the following approximate distribution function derived,

$$p(x) = \frac{6^6}{120} \left(\frac{x}{\langle x \rangle} \right)^5 e^{-6x/\langle x \rangle}, \tag{1}$$

where $p(x)$ is the probability of having a value x. This result is similar to the one derived from Mott [8] for the mass fragmentation

$$p(m) = \frac{e^{-(2m/\langle m \rangle)^{0.5}}}{\int_0^{+\infty} e^{-(2m/\langle m \rangle)^{0.5}} dm} \tag{2}$$

and rederived from Brostow and Rogers in 1985 [9] by using the theory of information. By using approximate methods we can easily derive the Voronoi diagrams and the related parameters and we can also explore intermediate situations in which the diagrams are not yet completely developed. This could be the case of foams in which the signals start from the various nuclei at shifted times and therefore generate a dynamical rather than a static situation.

Once we have obtained these results we shall limit our discussion to a qualitative level and we will not be able to produce approximate statistics because the resulting segments are not straight and their length is a function of the chosen time.

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2. The Two-Dimensional Algorithm

Given a certain value of pixels (which depends on the computer's memory) we introduce a two-dimensional lattice defined by *pixels* × *pixels* points $L_{k,m}$ that cover a square side × side. Consider a set of points r_l that are randomly distributed in the plane:

$$x_l = R[0, \text{side}],$$

$$y_l = R[0, \text{side}],$$

$$\vec{r}_l = x_l \vec{i} + y_l \vec{j}.$$

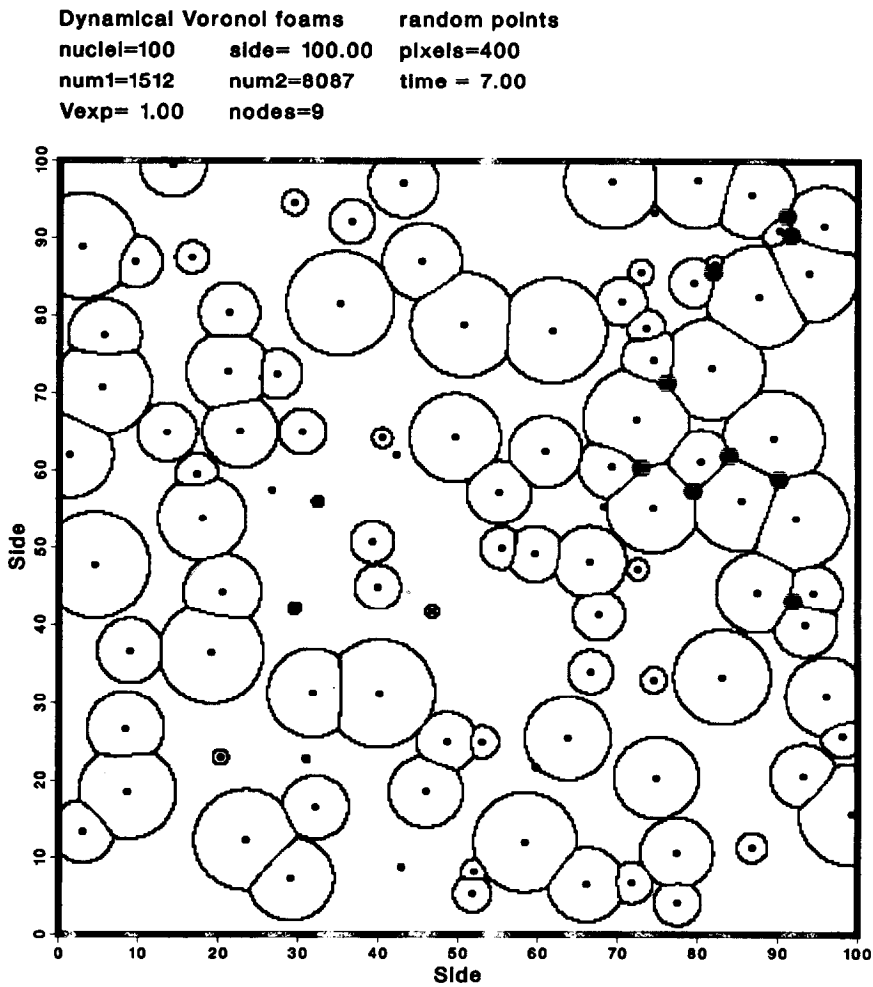

FIG. 1. The Voronoi diagram tessellations are drawn through little black squares. The nuclei are drawn through small black circles and the bigger circles mark the nodes.

These points become the centers of expansion of the signal at shifted times given by

$$t_j = R[0, \text{realtime}],$$

where realtime is the time at which we perform the analysis. In order to simplify the situation we assume a simple expansion law given by a linear time dependence,

$$\text{Radius}_j = F(\text{realtime} - t_j).$$

Due to the complexity of the problem we start with a first set of random points in the square, each point being generated at a different time. If an expanding circle reaches one of the nuclei we start again with a new set of points. The final distribu-

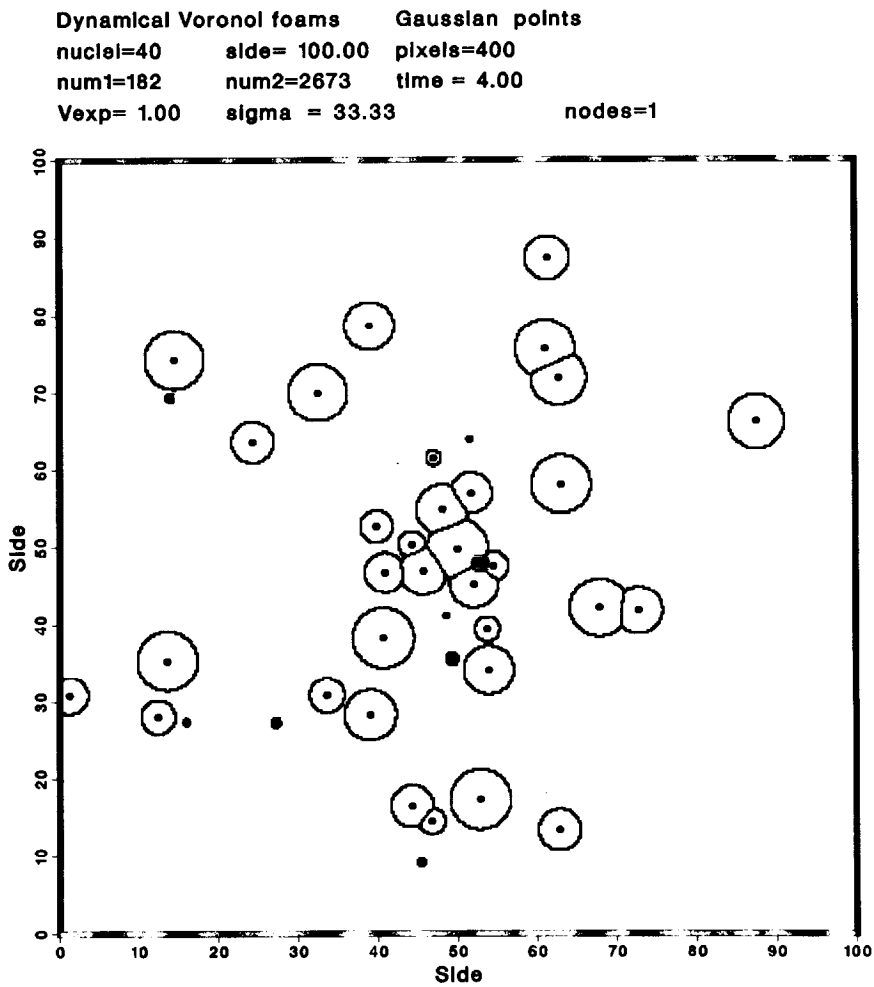

FIG. 2. The same as in Fig. 1 but with a Gaussian distribution of nuclei without marking the nodes.

tion will appear to be somewhat non-random in the sense that there are very few close first neighbours.

We introduce the concept of temporal distance t_d, that is, the time at which an expanding circle from the nucleus reaches the considered lattice point. We introduce the following definitions:

lines $\Rightarrow$ an element L_{km} belongs to a side of the dynamical polygons when the temporal distance from the two nearest nuclei (timely speaking) are greater than realtime and their difference is less than the time needed to cross the element.

nodes $\Rightarrow$ the same definition as for the lines but applied to the three nearest nuclei.

In order to complete the process we finally add the elements which have not yet collided.

We now illustrate the procedure with selected results. In Fig. 1 we show a typical situation in which the nuclei are randomly generated in the square and in Fig. 2 we show another one in which the nuclei are generated through a Gaussian distribution centered in the center of the square and with a dispersion as indicated in the figure. We note that now in both cases the expanding circles meet at curved lines rather than at straight ones and that this is due to the time delay of the signal from the various nuclei. This method is named the Johnson–Mehl, 1939, subdivision of space [10].

3. The Three-Dimensional Algorithm

The same technique can easily be applied to a space containing one more dimension. We now work on a three-dimensional lattice L_{kmn} to cover a cube with voxels.

The procedures to generate the points are similar to the ones of Section 2 and the faces, edges, and nodes are intersections of two, three, and four expanding spheres, respectively. We then add the points that are still expanding. A typical run shown in Figs. 3 and 4 represents the spatial distribution of the nuclei denoted by spheres at the end of little vertical sticks. In order to take into account the great variety of prospectives we introduce the observer's parameters: latitude, longitude, and distance.

Also, here three expanding spheres meet in curved lines (the edges) and two expanding shells generate (when they intersect) curved surfaces rather than planes.

4. Concluding Remarks

Thanks to the increased computing velocity (more MIPS) and the available dynamical memory (more Mb) we can easily divide the space in many little cubes and in every point to compute the time distance from the various nuclei. This allows us in a simple way to build approximate Voronoi diagrams. More detailed

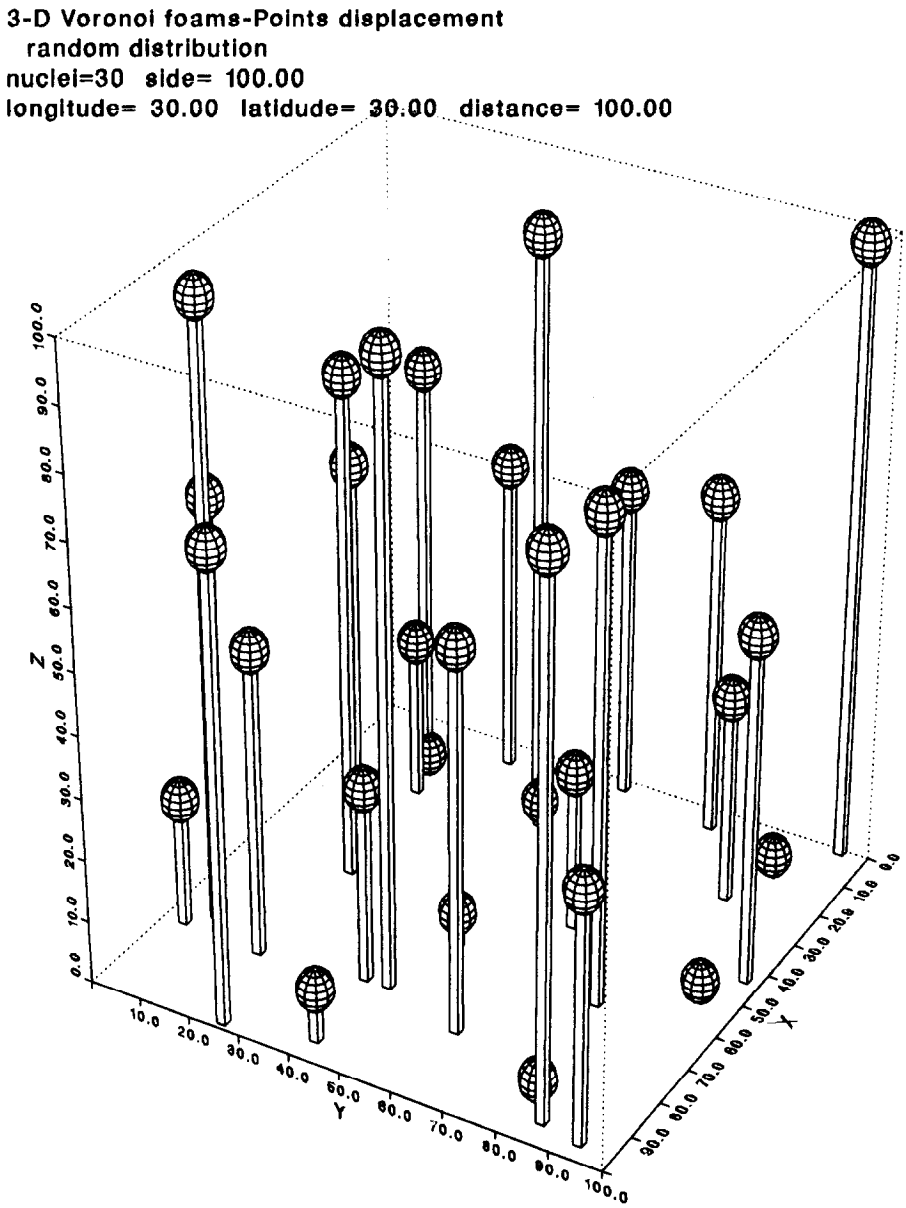

FIG. 3. The random distribution of nuclei in a cube.

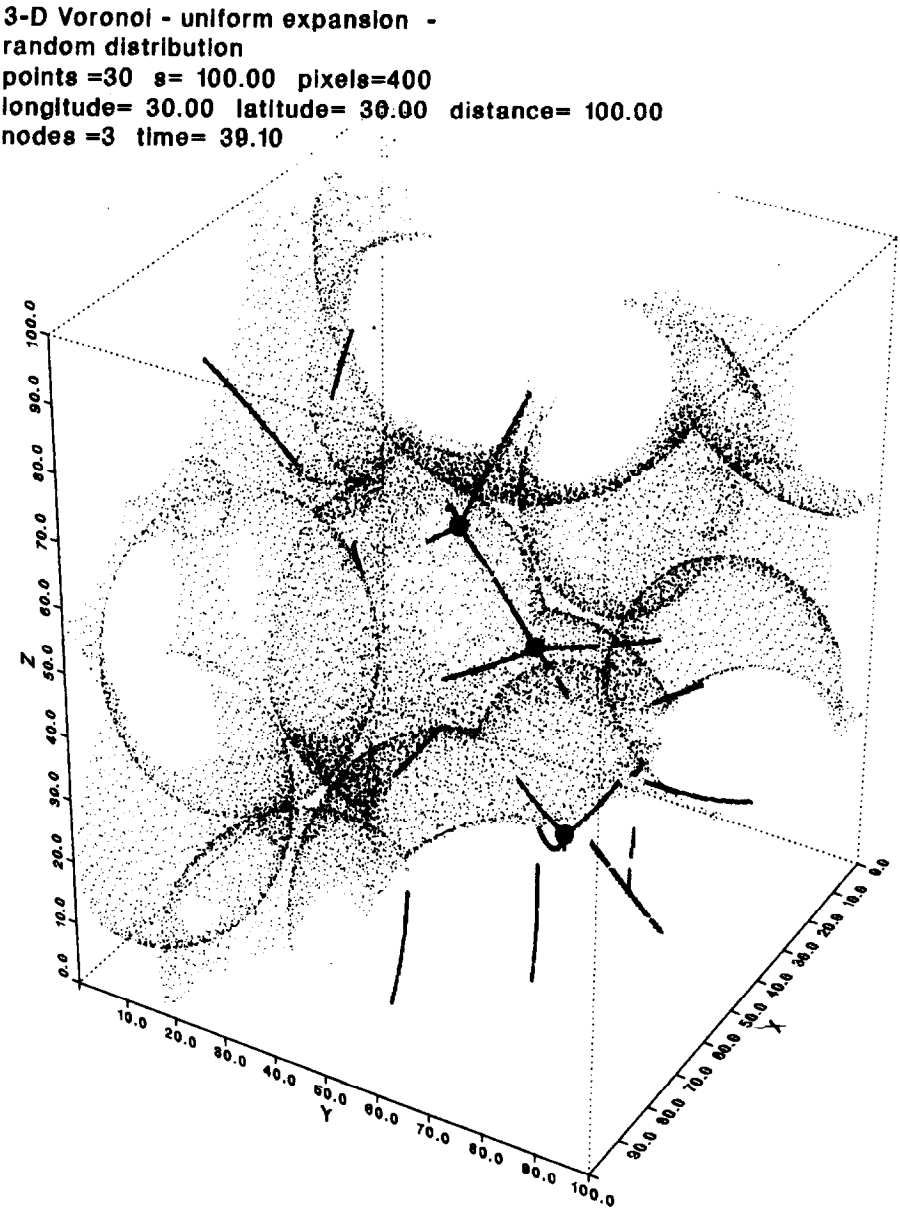

FIG. 4. The nodes, the edges, and the not yet collided shells are drawn with symbols of respectively decreasing size.

computations on the statistics of the obtained segments similar to those performed by Krishnamurthy, Brostow, and Sochanski [11] are in progress.

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